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Generalized Relativity: A Unified Field Theory Based on Free Geodesic Connections in Finsler Space

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An equation of structure on space-time is developed using the methods of Finsler geometry with an essentially unrestricted connection function, corresponding to a specification of the geodesic curves in the manifold. This contrasts with the usual approach in which the connection is derived in an explicit and restrictive manner from a metric. The equation of structure based on the "free" geodesic connection is found to incorporate, as a special case, equations which are closely comparable to both Einstein's equation of gravity in the presence of electromagnetic energy and Maxwell's equations for a chargeless electromagnetic field. Beyond this unification of electromagnetism and gravity, the theory appears to offer a wide scope for consideration of additional implications which may provide insights into other observed phenomena.

1. BACKGROUND

1.1. Introduction

This paper seeks to formulate the basis for a unified field theory using the methods of Finsler geometry. This objective has attracted the attention of a number of workers over a period of more than 50 years, dating back to Randers' (1941) original suggestion of a unified metric in the form

$$dl = \left(\sqrt{g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}} + A_i \frac{dx^i}{d\tau} \right) d\tau$$

Examples of authors who have addressed this subject include, in addition to Randers, Stephenson and Kilmister (1953) and Beil (1987, 1996).

Randers metric approaches are tantalizing, especially because the equation of a geodesic curve under a Randers metric closely matches the Lorentz

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equation for the motion of a charged particle in electromagnetic and gravitational fields. None of the work to date, however, has been able to achieve a complete and satisfactory unification of the field equations of gravity and electromagnetism on the basis of the Randers metric or any other Finsler approach. The present work may offer a step toward that objective.

The present theory begins with a *geodesic function*, that is, a function $B^k(x, y)$ which defines a collection of geodesic curves $C^k(\tau)$ in a manifold by specifying the "bend" $d^2C^k(\tau)/d\tau^2 = -B^k(C^i, dC^i/d\tau)$ of the geodesic which passes through each given point in each possible direction at that point. The geodesic function is not required to be defined uniquely by a particular metric, although it may be expected to be *consistent* with some metric in the sense that all geodesic curves defined by the geodesic function will be length-extremizing as measured under that metric. Starting from the geodesic function, the theory then proceeds in a straightforward way by defining a connection that is derived directly from the geodesic function, and using this connection to define a curvature tensor whose trace is set equal to zero to form the theory's equation of structure.

The free geodesic approach eliminates a significant constraint which has bound those theories which assume that the connection must be defined uniquely by a metric. Applying the results to geodesic functions that are consistent with the Randers metric, it is found that the equation of structure includes equations which are closely comparable to both Einstein's equation of gravity in the presence of electromagnetic energy and Maxwell's equations for a chargeless electromagnetic field.

1.2. Basic Concepts and Notation

Let M be a manifold. Given a point x in M, we denote the tangent space over x by T_xM. The tangent bundle of M, consisting of the collection of all of the T_xM with the canonical projection which carries every point in each T_xM onto x, is denoted by TM. Functions in Finsler geometry are frequently limited to the "punctured" or "slit" tangent bundle $TM\backslash\{0\}$ formed by removing the origin from each T_xM. The tangent sphere over x, S_xM, is the quotient space formed by identifying all points on each ray outward from the origin in the corresponding $T_xM\backslash\{0\}$, and the bundle formed from all of the S_xM is SM. Elements of TM may be referred to as "velocities," while elements of SM may be referred to as "directions."

Points in a Finsler space over space-time are represented by an octuple of real numbers (x^i, y^i), where the x^i represent the coordinates of a point in space-time and the $y^i = dx^i/d\tau$ represent the coordinates of a vector in the tangent space T_xM using the coordinate system which is induced by the coordinate system in M.

We define a *curve C* to be a subset of M which can be represented as the range of a smooth function $C^k(\tau)$ of a real parameter τ together with an orientation which is established by specifying a direction tangent to the curve at any point on the curve. A given curve can be represented by many different parametrizations. A "good" parametrization is one which is monotonically increasing in a direction which is consistent with the orientation of the curve (i.e., $dC^k/d\tau$ must have the same direction as the specified direction at the point where it is specified). Any good parametrization associates each point x on the curve with a point in $T_xM\backslash\{0\}$ by $(C^k(\tau)) \to (dC^k/d\tau)$. If a different parametrization is selected, each point is associated instead with $dC^k/d\sigma = (d\tau/d\sigma)\, dC^k/d\tau$, which lies on the same outward ray from the origin in T_xM. Thus we have a well-defined (for good parametrizations) association of each point x on the curve with a point in the corresponding S_xM, representing the direction of the curve at x.

We will designate derivatives with respect to the x coordinates by a subscript preceded by a comma:

$$f_{,i} = \frac{\partial f}{\partial x^i} \tag{1}$$

We will use the same notation to designate differentiation with respect to a scalar independent variable, such as a parameter, in which case the comma is placed before the name of the independent variable in the subscript:

$$C^k_{,\tau} = \frac{dC^k}{d\tau} \tag{2}$$

Many papers on Finsler geometry designate derivatives with respect to the y coordinates by plain, unadorned subscripts. This can sometimes lead to confusion when other subscripted (covariant) variables also appear. Accordingly, we will adopt a convention of designating y derivatives by a caret before the subscript:

$$f_{\wedge i} = \frac{\partial f}{\partial y^i} \tag{3}$$

A function f is said to be *homogeneous of degree n* if

$$F(x^i, my^i) = m^n F(x^i, y^i) \qquad \forall m > 0 \tag{4}$$

where n may be positive, zero, or negative. A function that is homogeneous of degree one is also simply referred to as a "homogeneous function."

The following relation, based on Euler's equation, applies to any differentiable function f of degree n:

$$f_{\wedge i}\, y^i = nf \tag{5}$$

2. GENERALIZED RELATIVITY

2.1. Geodesics and Connections

Typically, the definition of a connection, in both Riemannian and Finsler geometry, has begun with a metric which, together with its x and (in the Finsler case) y derivatives, is used to specify the connection. In the Finslerian case, several different connections have been identified, each with its own special characteristics. The principal metric-based connections are identified and discussed in Asanov (1985) and Bao *et al.* (1996). Geodesic curves $C^k(\tau)$ are derived from the connection by stipulating that the vector $C^k_{,\tau}$ will be carried along the curve by parallel transport, thus $C^k_{,\tau,\tau} = -\Gamma^k{}_{ij} C^i_{,\tau} C^j_{,\tau}$, where $C^k(\tau)$ is the geodesic and $\Gamma^k{}_{ij}$ is taken to represent the connection. We will refer to this metric-based approach, with its variety of connections, as the "standard approach."

As outlined in the Introduction, we shall follow a different tack, *beginning* with the geodesics. We define a *geodesic collection* as a set $\mathbb{C}$ of smooth curves on M satisfying the following requirement: for each $x \in M$ and for each $l \in S_x M$ there is one and only one curve $C \in \mathbb{C}$ which passes through x in direction l.

A *geodesic function* is defined as a smooth contravariant function $B^k(x^i, y^i)$ on $TM \backslash \{0\}$ which specifies the members of a geodesic collection by the relation

$$C^k_{,\tau,\tau}(\tau) = -B^k(C^i(\tau), C^i_{,\tau}(\tau)) \tag{6}$$

for each $x \in M$ and for each $y \in T_x M$ at that x. Clearly $C^k_{,\tau,\tau}(m\tau) = m^2 C^k_{,\tau,\tau}(\tau)$, so B^k is homogeneous of degree two. The geodesic function B^k is "free" in the sense that it is not restricted to any particular values derived from an initially selected metric. The only restriction will come from the equation of structure which we will develop.

The geodesic function is not uniquely associated with the geodesic collection because it is dependent on the parametrization of the geodesics. For any geodesic function $C^k(\tau)$, if we replace the parameter τ with $\sigma(\tau)$, we have

$$\begin{aligned} C^k_{,\tau,\tau} &= -B^{(\tau)k}(C^i, C^i_{,\tau}) \\ &= C^k_{,\sigma,\sigma}\sigma^2_{,\tau} + C^k_{,\sigma}\sigma_{,\tau,\tau} \\ &= -B^{(\sigma)k}(C^i, C^i_{,\sigma})\sigma^2_{,\tau} + C^k_{,\sigma}\sigma_{,\tau,\tau} \end{aligned} \tag{7}$$

where $B^{(\tau)k}$ and $B^{(\sigma)k}$ refer to the geodesic functions generated by the τ- and

σ-parametrizations, respectively. Since B^k is homogeneous of degree two and $C^i_{,\tau} = C^i_{,\sigma}\sigma_{,\tau}$, we have, evaluated at $(C^i, C^i_{,\sigma})$,

$$B^{(\tau)k} = B^{(\sigma)k} - \frac{\sigma_{,\tau,\tau}}{\sigma^2_{,\tau}} y^k \tag{8}$$

We define the connection by

$$B^k{}_i = B^k_{\wedge i} \tag{9}$$

Also,

$$B^k{}_{ij} = B^k{}_{i\wedge j} \tag{10}$$

For an arbitrary function f defined on $TM\backslash\{0\}$, the tensor derivative based on the free geodesic connection, which we will designate by a star (*) before the index, is

$$f_{*i} = f_{,i} - \tfrac{1}{2} f_{\wedge m} B^m{}_i \tag{11}$$

The tensor character of f_{*i} can be verified as follows. Let $\{x^i\}$ and $\{x^{i'}\}$ represent two sets of coordinates on M, which generate coordinates $\{y^i\}$ and $\{y^{i'}\}$, respectively, on the T_xM. We then have

$$\begin{aligned} f_{,i'} &= f_{,i} x^i_{,i'} + f_{\wedge i} y^i_{,i'} \\ &= f_{,i} x^i_{,i'} + f_{\wedge i}(y^{j'} x^i_{,j'})_{,i'} \\ &= f_{,i} x^i_{,i'} + f_{\wedge i} x^i_{,i',j'} y^{j'} \end{aligned} \tag{12}$$

On the other hand, it is clear that

$$\begin{aligned} f_{\wedge m'} &= f_{\wedge m} y^m_{\wedge m'} + f_{,m} x^m_{\wedge m'} \\ &= f_{\wedge m}(y^{n'} x^m_{,n'})_{\wedge m'} \\ &= f_{\wedge m} \delta^{n'}_{m'} x^m_{,n'} \\ &= f_{\wedge m}\, x^m_{,m'} \end{aligned} \tag{13}$$

where $\delta^{n'}_{m'}$ is the Kronecker delta. The second term on the first line and the y derivative of the second factor in parentheses in the second line are both zero because a change in a vector in T_xM (whether measured by primed or unprimed coordinates) does not change the underlying point $x \in M$. Finally, for the variance of the geodesic function, we have

$$\begin{aligned} B^{k'} &= -C^{k'}_{,\tau,\tau} \\ &= -(C^k_{,\tau} x^{k'}_{,k})_{,\tau} \end{aligned}$$

$$= -C^k_{,\tau,\tau} x^{k'}_{,k} - C^k_{,\tau} x^{k'}_{,k,i} C^i_{,\tau}$$
$$= B^k x^{k'}_{,k} + x^{k'}_{,k} x^k_{,i',j'} y^{i'} y^{j'} \tag{14}$$

where in the final equality we have made use of $0 = (\delta^{k'}_{j'})_{,i'} = (x^{k'}_{,k} x^k_{,j'})_{,i'} = x^{k'}_{,k,i} x^k_{,j'} x^i_{,i'} + x^{k'}_{,k} x^k_{,i',j'}$, therefore (multiplying by $x^{i'}_{,m} x^{j'}_{,n}$ and changing indices), $x^{k'}_{,k,i} = -x^{k'}_{,m} x^m_{,i',j'} x^{i'}_{,i} x^{j'}_{,k}$. Putting these results all together, we have

$$\begin{aligned} f_{*i'} &= f_{,i'} - \tfrac{1}{2} f_{\wedge k'} B^{k'}_{\wedge i'} \\ &= f_{,i} x^i_{,i'} + f_{\wedge i} x^i_{,i',j'} y^{j'} - \tfrac{1}{2} f_{\wedge k} x^k_{k'} (B^p x^{k'}_{,p} + x^{k'}_{,m} x^m_{,m',n'} y^{m'} y^{n'})_{\wedge i'} \\ &= f_{,i} x^i_{,i'} - \tfrac{1}{2} f_{\wedge k} B^k_{\wedge i} x^i_{i'} \\ &= f_{*i} x^i_{i'} \end{aligned} \tag{15}$$

where, in the second line, the second term cancels with the second term in parentheses, eliminating the unwanted variance.

Tensor derivatives of functions with covariant and/or contravariant indices necessarily involve additional terms similar to those found in Riemannian analysis. Thus the general definition of a tensor derivative using the free geodesic connection B^k is

$$\begin{aligned} (f^{i_1..i_a}_{j_1..j_b})_{*m} = (f^{i_1..i_a}_{j_1..j_b})_{,m} - \tfrac{1}{2} (f^{i_1..i_a}_{j_1..j_b})_{\wedge k} B^k{}_m \\ + \tfrac{1}{2} (f^{ki_2..i_a}_{j_1..j_b}) B^{i_1}{}_{km} + .. + \tfrac{1}{2} (f^{i_1..i_{a-1}k}_{j_1..j_b}) B^{i_a}{}_{km} \\ - \tfrac{1}{2} (f^{i_1..i_a}_{kj_2..j_a}) B^k{}_{j_1 m} - .. - \tfrac{1}{2} (f^{i_1..i_a}_{j_1..j_{b-1}k}) B^k{}_{j_b m} \end{aligned} \tag{16}$$

2.2. Curvature, and Equation of Structure

It can be shown that for any f,

$$f_{*i*j} - f_{*j*i} = -\tfrac{1}{2} f_{\wedge k} S^k{}_{ij} \tag{17}$$

where $S^k{}_{ij}$ is a homogeneous function of degree 2 defined by

$$S^k{}_{ij} = B^k{}_{i,j} - B^k{}_{j,i} - \tfrac{1}{2} B^k{}_{im} B^m{}_j + \tfrac{1}{2} B^k{}_{mj} B^j{}_i \tag{18}$$

$S^k{}_{ij}$ is therefore a tensor on $TM \backslash \{0\}$, and naturally suggests itself as the curvature tensor associated with the geodesic function B^k.

Setting the trace of the curvature tensor to zero, we have the following as our proposed equation of structure:

$$\begin{aligned} S_i &= S^j{}_{ij} \\ &= B^j{}_{i,j} - B^j{}_{j,i} - \tfrac{1}{2}B^j{}_{im}B^m{}_j + \tfrac{1}{2}B^j{}_{jm}B^m{}_i \\ &= 0 \end{aligned} \tag{19}$$

We can replace (19) with two somewhat simpler equations:

$$\tfrac{1}{2}S_i y^i = B^j{}_{,j} - \tfrac{1}{2}B^j{}_{j,i}y^i - \tfrac{1}{4}B^j{}_m B^m{}_j + \tfrac{1}{2}B^j{}_{jm}B^m = 0 \tag{20}$$

$$S_i - (\tfrac{1}{2}S_j y^j)_{\wedge i} = \tfrac{1}{2}(B^j{}_{ji,k}y^k - B^j{}_{j,i} - B^j{}_{ji\wedge m}B^m) = 0 \tag{21}$$

Defining $B = B^j{}_j$, we can write (21) as follows:

$$B_{\wedge i,k}y^k - B_{,i} - B_{\wedge i\wedge k}B^k = 0 \tag{22}$$

2.3. Consistent Metrics

Up to this point, we have worked without any metric. Let us assume now that there is a homogeneous function F, the metric, on $TM\backslash\{0\}$, and define $F_i = F_{\wedge i}$ and $F_{ij} = F_{i\wedge j}$. We assume also that all of the geodesics in the geodesic collection represented by B^k are locally length-extremizing under the metric F, i.e.,

$$\delta \int_{\tau_0}^{\tau_1} F(C^i(\tau), C^i_{,\tau}(\tau))\, d\tau = 0 \tag{23}$$

It is well known that this local extremization implies the following relationship between F and each geodesic curve $C^k(\tau)$:

$$F_{ik}C^k_{,\tau,\tau} + F_{i,k}y^k - F_{,i} = 0 \tag{24}$$

In view of (6), we have

$$F_{ik}B^k = F_{i,k}y^k - F_{,i} \tag{25}$$

We will say that a metric that satisfies (25) is *consistent* with the geodesic function B^k.

The consistency relationship between geodesic curves and geodesic functions is not one-to-one. Because F_i is homogeneous of degree zero, then $F_{ik}y^k = 0$, and B^k may thus be changed by the addition of any term of the form κy^k (κ an arbitrary homogeneous function) without changing the result. This transformation corresponds to the transformation in the geodesic function that results from a change in the parameters of the geodesic curves as shown in (8), with $\kappa = -\sigma_{,\tau,\tau}/\sigma^2_{,\tau}$.

We can apply these principles to define an equivalence relationship among geodesic functions which are consistent with the same metric, as follows:

$$B^k \sim B^k + \kappa y^k \tag{26}$$

where κ is an arbitrary homogeneous function on $TM\backslash\{0\}$. We can use this relation to establish a technique for moving from the metric to a consistent geodesic function, which will be particularly useful in dealing with the Randers metric.

Given a metric F, F_{ij} is a homogeneous function of degree negative one. Assume that there is a homogeneous function h^{jk}, symmetric in the j and k indices, that satisfies

$$h^{jk}F_{ij} = \delta_i^k + \theta_i y^k \tag{27}$$

where θ_i is an arbitrary indexed function, homogeneous of degree negative one.

It is easy to see that (27) can be satisfied in any case where the metric function $g_{ij} = \frac{1}{2}(F^2)_{\wedge i \wedge j}$ has a nonzero determinant. In this case, we can define g^{jk} by $g^{jk}g_{ij} = \delta_i^k$, and then (27) will be satisfied by $h^{jk} = Fg^{jk}$, just as one example:

$$\begin{aligned} Fg^{jk}F_{ij} &= Fg^{jk}(\sqrt{F^2})_{\wedge i \wedge j} \\ &= Fg^{jk}\left(\frac{(F^2)_{\wedge i \wedge j}}{2F} - \frac{(F^2)_{\wedge i}(F^2)_{\wedge j}}{4F^3}\right) \\ &= Fg^{jk}\left(\frac{g_{ij}}{F} - \frac{F_i}{F^2}\, g_{jm}y^m\right) \\ &= \delta_i^k - \frac{F_i}{F}\, y^k \end{aligned} \tag{28}$$

Thus (27) is satisfied with $\theta_i = -F_i/F$. In contrast to g^{jk}, h^{jk} is not well defined by (27), since (27) is also solved by any $\tilde{h}^{jk} = h^{jk} + \zeta^j y^k + \zeta^k y^j$, where ζ^j is an arbitrary indexed function, homogeneous of degree negative one.

We now have a collection of solutions to (25) if we define

$$B^k = h^{jk}(F_{j,m}y^m - F_{,j}) + \kappa y^k \tag{29}$$

because then

$$\begin{aligned} F_{ik}B^k &= F_{ik}(h^{jk}(F_{j,m}y^m - F_{,j}) + \kappa y^k) \\ &= (\delta_i^j + \theta_i y^j)(F_{j,m}y^m - F_{,j}) \\ &= F_{i,m}y^m - F_{,i} \end{aligned} \tag{30}$$

Comparing (25) with (22), it is easy to see that (22) is satisfied if we have B^k consistent with F and

$$B = \lambda F + \phi_{,i} y^i \tag{31}$$

where λ is a "universal" constant and ϕ is a function of x only. It is not clear whether (31) is also a *necessary* condition to B's satisfaction of (22), but we will use this as a working assumption. It should be noted that the term $\phi_{,i} y^i$ reflects a gauge freedom in F in that it can be incorporated in F without affecting the calculation in (25). This term therefore does not have to be expressed separately in (31), but it will prove useful for future purposes to retain it. Taking this into account, we see that, in the case of metric-compatible geodesic functions, we can reduce the equation of structure to two equations—(20) and (31)—in two independent variables—F and κ, which are together sufficient to specify B^k employing (27) and (29).

It is also interesting to note that if we begin with a geodesic function that satisfies the equation of structure, and if $\lambda \neq 0$, then we can determine a consistent metric from the geodesic function by simply setting $F = B$. In this case, (25) is clearly satisfied by virtue of (22).

2.4. Comparison of Approaches

Now that we have laid out the basic elements of the free geodesic theory, it is time to pause to reflect on its significance in relation to the standard approach to Finsler analysis.

The real point of departure comes in the different responses to (25). This equation defines the relationship between the metric function and the geodesic function, but, given a metric, the metric does not uniquely define the geodesic function or the connection, either in the Finsler context or in the Riemannian context. Versions of the standard approach, however, uniformly start from the premise that such a correspondence must be imposed in some way so that there will be a unique, well-defined connection function derived from the metric. Thus, in the Riemannian context, the standard formulation of general relativity assumes that the connection must be affine, which forces a one-to-one correspondence with the metric.

In Finsler geometry, the tendency has been to work with methods that stick closely to the techniques that have proved successful in the Riemannian context. Even though Finsler geometry is distinguished by the fact that the metric can be expressed as a scalar, $F(x, y)$, the focus in building connections has been on the covariant metric function $g_{ij} = \frac{1}{2}(F^2)_{\wedge i \wedge j}$ rather than on F itself. [Beil's (1996) theory even starts with a covariant metric function which is not derived from F, and is not a y derivative of any function.] g_{ij} looks, and to some extent behaves, like the Riemannian metric function. In particular, it is (unlike F_{ij}) generally amenable to the definition of a contravariant counterpart g^{jk} via the requirement $g^{jk} g_{ij} = \delta_i^k$, enabling the creation of a "natural" geodesic function satisfying (25) by a formula which is almost identical to that for the affine connection in Riemannian geometry:

$$G^k = \tfrac{1}{2} g^{km}(g_{im,j} + g_{jm,i} - g_{ij,m})\, y^i y^j \tag{32}$$

This geodesic function can be used to define a connection $G^k{}_{ij} = G^k_{\wedge i \wedge j}$, the Berwald connection. For geometers, the Berwald connection is not totally satisfying because it is neither metric-compatible (the tensor derivative of the metric function is not identically zero) nor torsion-free (the order of the indices affects the result in double tensor differentiation of a scalar function). By adding extra terms, it is possible to create variants that are either metric-compatible (the Cartan connection) or torsion-free (the Chern connection). These connections have been found useful in a variety of geometric studies, but have not found any application to physical science.

From a philosophical standpoint, however, there is much to be said for the Finsler approach to formulation of a unified field theory. As Riemann himself acknowledged, the limitation of standard Riemannian analysis to a quadratic form of metric is essentially arbitrary, reflecting principally the need for computational simplicity rather than any universal principle. There does not appear to be any *a priori* reason why we should assume that the structure of the universe is so simple that distances in space-time can always be measured in this particular way. Or, put another way, it seems that there ought to be some way of extending the success of general relativity to a more broadly defined metric.

If we are to make another try at a physically meaningful Finsler theory, the idea of basing it on a geodesic function rather than on a metric also has philosophical appeal. A geodesic, viewed as a collection of points, represents a tangible object in space-time, and a geodesic collection—which is what the geodesic function represents—seems to offer a very fundamental way of expressing the "self-connectedness" of space-time. A metric, on the other hand, is an abstract concept, created by an observer for the purpose of describing and analyzing the objects he observes. If a theory is to capture the structure of space-time on anything like an "absolute" basis, it seems that it should start with something that is integrally a part of space-time, not with some abstraction. This perspective takes on greater importance when we confront the fact that standard Finsler geometry offers a plethora of connections, with no solid basis for choosing one over another.

To be sure, the geodesic function contains an arbitrary aspect, that of parametrization, in addition to its description of the geodesic collection. We may, however, relate the theory directly to the geodesic collection by viewing the geodesic function as a representative of an equivalence class of geodesic functions as defined in (26). Similarly, we define equivalence classes of curvature tensors,

$$S^k{}_{ij} \sim S^k{}_{ij} + (\kappa_{\wedge i * j} - \kappa_{\wedge j * i}) y^k + \delta^k_i (\kappa_{* j} - \tfrac{1}{2}\kappa\kappa_{\wedge j}) - \delta^k_j (\kappa_{* i} - \tfrac{1}{2}\kappa\kappa_{\wedge i}) \tag{33}$$

and of curvature traces,

$$S_i \sim S_i + \kappa_{\wedge i * j} y^j - 4\kappa_{* i} + \tfrac{3}{2}\kappa\kappa_{\wedge i} \tag{34}$$

where κ in each case represents an arbitrary scalar function on $TM \backslash \{0\}$. We now recognize that the appropriate measure of the curvature of M is an *equivalence class* of tensors on $TM \backslash \{0\}$ in the form $S^k{}_{ij}$ as defined by (33), and not the tensor itself. Two curvature tensors which fall within the same equivalence class must be regarded as representing the same physical curvature. This interpretation seems virtually unavoidable given that otherwise we could have two manifolds all of whose geodesic curves (viewed as sets of points) are identical, but which have different curvatures at all points. Making use of the equivalence class of curvature traces, we can now write the equation of structure in the form

$$[S_i] = [0] \tag{35}$$

where the brackets designate an equivalence class as defined in (34).

This approach addresses to some extent Bao *et al.*'s (1996) call for a focus on SM as one of the principal entities of Finsler geometry. The equivalence classes of geodesic functions effectively measure the changing of the *direction* (a vector in SM) of a tangent to the curve as one traces along its length, without regard to the particular *velocities* (vectors in TM) that arise from particular parametrizations of the curve. On the other hand, we cannot limit our attention to tensors defined on SM, because the equivalence classes begin to take on a more complex structure as the analysis proceeds.

It should be noted that we have gone straight from the equivalence classes of geodesic functions to the equivalence classes of curvature tensors without attempting to derive the latter from any equivalence classes of tensor derivatives, as in (11) and (17). It is easy enough to define equivalence classes of tensor derivatives based on the equivalence classes of geodesic functions, but if this were then applied in (17) we would find an excessive amount of variance in the definition of the curvature tensor since we would have to use a distinct variance function in lieu of κ for each time B^κ appears in the formula. Evidently the notion of curvature and the equation of structure are tied much more closely to the geodesic collection and are not dependent on the notion of a tensor derivative.

In any event, the geodesic collections on M stand in a one-to-one relationship with the equivalence classes $[B^k]$ of geodesic functions, and the $[S_i]$ represent a well-defined functional of the $[B^k]$. Thus the equivalence-based equation of structure, (35), reflects in a well-defined manner the inherent structure of M as embodied in the geodesic collection. By focusing on the equivalence classes of functions related to the geodesic collection, we have eliminated the apparent arbitrariness involved in the choice of parametrization of the geodesics.

It was suggested earlier in this section that the objective of this work is to find "some way of extending the success of general relativity to a more broadly defined metric." Based on the results to be shown in Section 3, we believe that this objective may well have been achieved. Accordingly, we propose to refer to the free geodesic theory and its equation of structure, (19), by the name "generalized relativity."

3. APPLICATION TO THE RANDERS METRIC

3.1. Preliminaries

As presented, the equation of structure of generalized relativity is not limited to any particular form of metric. On the other hand, any attempt to find general solutions must confront a great deal of inherent complexity, and it is not easy to see how the equation might reflect the physical world without reducing it to a form that can be expressed and understood in terms of Riemannian space. We therefore move to a consideration of how the equation of structure applies in the case of a metric that is limited to the Randers form. In taking this step, however, we must emphasize that we are not limiting in any way the scope of the equation of structure developed in Section 2. We hold to the proposition that (19) provides a complete description of the structure of space-time. This section presents only the results of applying that description to a narrow, but highly salient, class of possible metrics and geodesic functions from among all those that are available on $TM\backslash\{0\}$.

Thus we assume

$$F = g + 2A_i(x)y^i \tag{36}$$

where we define

$$g = \sqrt{-g_{ij}(x)y^i y^j} \tag{37}$$

Throughout this section, g_{ij} and g^{jk} will be used to refer to the standard Riemannian metric and its contravariant counterpart, not the Finslerian versions. Also, we will use g_{ij} and g^{jk} to raise and lower indices in the standard way. Later in this section we will use the standard notation $F_{ij} = A_{i,j} - A_{j,i}$, which should not be confused with the Finsler function $F_{ij} = F_{\wedge i \wedge j}$ as used previously and in Section 3.2. The coefficient in the A term in (36) and the negative sign within the square root in (37) have been included to assure that the results will match to standard scaling and sign conventions, as will be seen.

3.2. Determining h^{jk}

Calculating the second y-derivatives of (36), the y-linear term $A_i y^i$ is eliminated, and we have

$$F_{ij} = -\frac{g_{ij}}{g} - \frac{g_{im}g_{jn}y^m y^n}{g^3} \tag{38}$$

We define

$$h^{jk} = -gg^{jk} \tag{39}$$

Then

$$\begin{aligned} h^{jk}F_{ij} &= -gg^{jk}\left(-\frac{g_{ij}}{g} - \frac{g_{im}g_{jn}y^m y^n}{g^3}\right) \\ &= \delta_i^k + \frac{g_{im}y^m y^k}{g^2} \end{aligned} \tag{40}$$

Thus h^{jk} satisfies (27) with $\theta_i = g_{\mathrm{im}}y^m/g^2$.

3.3. The Geodesic Function

Substituting (36) and (39) in (29), we then have

$$B^k = \Gamma^k{}_{ij}y^i y^j - 2F^k{}_j y^j g + \kappa' y^k \tag{41}$$

where

$$\Gamma^k{}_{ij} = \tfrac{1}{2}g^{km}(g_{mi,j} + g_{mj,i} - g_{ij,m}) \tag{42}$$

$$F_{ij} = A_{i,j} - A_{j,i} \tag{43}$$

and

$$\kappa' = \kappa - \frac{g_{mn,p}y^m y^n y^p}{2g^2} \tag{44}$$

is a function on $TM\backslash\{0\}$.

We may contrast this very streamlined geodesic function with the difficulties that are encountered in an attempt to apply the standard Finslerian approach to the Randers metric. For purposes of this subsection, we will use $\mathbf{g}_{ij}$ and $\mathbf{g}^{jk}$ to represent the Finsler metric function and its contravariant counterpart as distinguished from the Riemannian versions; thus $\mathbf{g}_{ij} = \frac{1}{2}(F^2)_{\wedge i \wedge j}$ and $\mathbf{g}^{jk}\mathbf{g}_{ij} = \delta_i^k$. In the standard approach, first, we find that $\mathbf{g}^{jk}$ is essentially impossible to express explicitly in terms of the Riemannian elements of the Randers metric because of the extra terms encountered in

$$\mathbf{g}_{ij} = -g_{ij} + 4A_iA_j - 2(g_{im}A_j + g_{jm}A_i + g_{ij}A_m)\frac{y^m}{g} - 2g_{im}g_{jn}A_p\frac{y^my^ny^p}{g^3} \tag{45}$$

A second problem comes in working out the expression within parentheses in the definition of G^k, at (32), which results in a bewildering array of terms. We can try to reduce this to its bare electromagnetic essentials by assuming constant g_{ij} and disregarding second-order terms in A_i and its derivatives, which yields

$$\begin{aligned} G^k &= \tfrac{1}{2}\mathbf{g}^{km}(\mathbf{g}_{im,j} + \mathbf{g}_{jm,i} - \mathbf{g}_{ij,m})y^iy^j \\ &= \tfrac{1}{2}\mathbf{g}^{km}((F^2)_{\wedge m,i}y^i - (F^2)_{,m}) \\ &\approx -2g^{ki}F_{ij}y^jg + 2A_{i*j}\frac{y^iy^jy^k}{g^3} \end{aligned} \tag{46}$$

The extra term on the right stands in the way of any attempt to create a physically meaningful field equation out of the Randers metric using the standard approach. Both of these issues are resolved directly and without any contrivance in the free geodesic approach.

3.4. Calculating and Simplifying κ'

Based on (41) and the working assumption referred to at (31) above, and assuming here and for the balance of this paper that M has four dimensions, we have

$$\begin{aligned} B &= 2\Gamma^k{}_{ik}y^i + 5\kappa' \\ &= (\ln(-\det(g_{jk})))_{,i}y^i + 5\kappa' \\ &= \lambda(g + 2A_iy^i) + \phi_{,i}y^i \end{aligned} \tag{47}$$

For present purposes, we will assume that λ is zero or negligibly small, and we will thus ignore it in order to show the resulting equations most simply. We then have

$$\kappa' = \frac{1}{5}(\phi - \ln(-\det(g_{jk})))_{,i}y^i \tag{48}$$

We now define

$$K = \exp\left(-\frac{1}{10}(\phi - \ln(-\det(g_{jk})))\right) \tag{49}$$

which is a Riemannian variable. Then, combining (41), (48), and (49), we have

$$B^k = \Gamma^k{}_{ij}\, y^i y^j - 2F^k_j\, y^j g - 2\frac{K_{,i}}{K}\, y^i y^k \tag{50}$$

3.5. Randers-Metric Equation of Structure

Substituting (50) in (20), we complete the equation of structure for the Randers metric, assuming $\lambda = 0$, resulting in the following:

$$\begin{aligned} \tfrac{1}{2} S_i y^i &= \left(R_{ij} - 2F_{im}F_j{}^m - g_{ij}F_{mn}F^{mn} + 3\,\frac{K_{,i;j}}{K} \right) y^i y^j \\ &\quad + 2\left(F_i{}^m{}_{;m} - 3\,\frac{K_{,m}}{K}\, F_i{}^m \right) y^i g \\ &= 0 \end{aligned} \tag{51}$$

where

$$R_{ij} = \Gamma^k{}_{ij,k} - \Gamma^k{}_{ik,j} - \Gamma^k{}_{im}\Gamma^m{}_{ik} + \Gamma^k{}_{mk}\Gamma^m{}_{ij} \tag{52}$$

is the Ricci tensor of general relativity, and a semicolon before an index in a subscript indicates a tensor derivative calculated using the gravitational metric g_{ij}.

Since (51) must be true across all of $TM\backslash\{0\}$, i.e., for all (y^i) and accordingly for all g, it gives us two separate equations on M corresponding to the fundamental equation:

$$R_{ij} - 2F_{im}F_j{}^m - g_{ij}F_{mn}F^{mn} + 3\,\frac{K_{,i;j}}{K} = 0 \tag{53}$$

$$F_i{}^m{}_{;m} - 3\,\frac{K_{,m}}{K}\, F_i{}^m = 0 \tag{54}$$

3.6. Results of Applying the Contracted Bianchi Identity

Applying the contracted Bianchi identity to (53), after some manipulation and substitutions based on (53) and (54), we find the following:

$$\begin{aligned} &\left(F_{mn}F^{mn} + g^{mn}\,\frac{K_{,m;n}}{K} - 4g^{mn}\,\frac{K_{,m}K_{,n}}{K^2} \right)_{;i} \\ &\quad + 2\,\frac{K_{,i}}{K}\left(F_{mn}F^{mn} + g^{mn}\,\frac{K_{,m;n}}{K} - 4g^{mn}\,\frac{K_{,m}K_{,n}}{K^2} \right) = 0 \end{aligned} \tag{55}$$

Thus

$$F_{mn}F^{mn} + g^{mn}\frac{K_{,m;n}}{K} - 4g^{mn}\frac{K_{,m}K_{,n}}{K^2} = \frac{\Phi}{K^2} \tag{56}$$

where Φ is a "universal" constant.

We can use this result in making a preliminary evaluation of whether the system represented by (53) and (54) has an initial value formulation. In the case of the vacuum Einstein equation of general relativity, the contracted Bianchi identity reveals four interdependencies among the 10 field equations, leaving 6 independent equations. Also in that case, the 10 variables in the components of g_{ij} reflect only 6 true degrees of freedom in the physical description because of the gauge invariances related to transformations in the four-dimensional coordinate system. With 6 independent equations and 6 "nongauge" variables, the vacuum Einstein equation is thus open to an initial value formulation, as discussed in Wald (1984). In (53) and (54), the contracted Bianchi identity represents just 3 interdependencies, since the four equations of the identity are reduced to one nontrivial equation, (56). An additional interdependency arises from (54) as follows: define $\mathfrak{F}_i = F_i{}^m{}_{;m} - 3(K_{,m}/K)F_i{}^m$; we then have, as an identity, $(\mathfrak{F}_i/K^3)_{;i} \equiv 0$, which establishes the interdependency. Thus in this case we have 10 independent equations [14 basic equations in (53) and (54) minus a total of 4 interdependencies] and 10 "nongauge" variables (10 components of g_{ij} plus 4 components of A_i plus 1 component of K, minus 4 degrees of coordinate gauge freedom in the g_{ij} minus 1 degree of Lorentz gauge freedom in the A_i). Thus, although it might seem at first that the introduction of K would leave the system underdetermined, the free geodesic equation of structure under the Randers metric with $\lambda = 0$ appears also to be open to an initial value formulation.

3.7. Reformulation

Looking at the two equations developed in Section 3.5, we see that:

First, (54) is nearly identical to Maxwell's vacuum electromagnetic field equations, but for the second term on the left. Assuming that $K_{,m}/K$ is very small in the region under observation, the effect of this term may not be measurable in laboratory experiments, which would leave us with an equation that is essentially equivalent to Maxwell's.

Second, (53) is equivalent to the Einstein gravitation equation with an energy field

$$T_{ij} = 2F_{im}F_j{}^m - 2g_{ij}F_{mn}F^{mn} - 3\frac{K_{,i;j}}{K} + \frac{3}{2}g_{ij}g^{mn}\frac{K_{,m;n}}{K} \tag{57}$$

(57) is not quite as we could expect it, because of the unusual relationship

of the terms in the electromagnetic components of the total energy, while the nature of the remaining terms is unclear. We can make this look more familiar by rewriting (53) using a substitution based on (56) and expressing it in terms of the Einstein tensor, as follows

$$\begin{aligned} G_{ij} &= R_{ij} - \tfrac{1}{2} g_{ij} g^{mn} R_{mn} \\ &= 8\pi (T_{(E)ij} + T_{(K)ij}) \end{aligned} \tag{58}$$

where

$$T_{(E)ij} = \frac{1}{4\pi} \left(F_{im} F_j{}^m - \tfrac{1}{4} g_{ij} F_{mn} F^{mn}\right) \tag{59}$$

$$T_{(K)ij} = \frac{3}{16\pi} \left(-2 \frac{K_{i;j}}{K} + g_{ij} \left(2g^{mn} \frac{K_{,m;n}}{K} - 4g^{mn} \frac{K_{,m} K_{,n}}{K^2} - \frac{\Phi}{K^2}\right)\right) \tag{60}$$

and

$$\begin{aligned} T_{(E)i}{}^{m}{}_{;m} &= -T_{(K)i}{}^{m}{}_{;m} \\ &= \frac{3}{4\pi} \left(\frac{K_{,m}}{K} F_{in} F^{mn}\right) \end{aligned} \tag{61}$$

In this formulation, we see that the equation of structure as applied to the Randers metric is equivalent to Einstein's equation of gravity in the presence of electromagnetic energy, together with an additional energy field $T_{(K)ij}$. The two energy fields interact with the force shown in (61). The substitution that we have made may be justified on the ground that it divides the total energy into components, based separately on the electromagnetic and K fields, with the simplest possible expression for the interaction force. Furthermore, in the spherically symmetric situation at least, as we will show in Section 4.1.3, this interaction force is smaller than it would be without the substitution. Quite apart from considerations of familiarity, these findings support the reformulation of (57) in the manner of (58)–(60).

If the K field were constant everywhere, clearly (54) and (58)–(60) would match very closely to existing theory. On the other hand, (56) indicates that the electromagnetic field must generate some variance in K (except in cases where $F_{mn}F^{mn} = 0$. Preliminary consideration of issues relating to the structure of solutions of (53) and (54) and their consistency with astronomical observations suggests that this effect is limited to the immediate region of the electromagnetic field, and will not carry over into regions where the electromagnetic field is neutralized—at least not in the real world, although it might be permitted under the equations. This analysis, which will not be pursued in detail in this paper, is quite uncertain at this stage, and the nature and effect of the K field clearly require further study.

4. PRELIMINARY INVESTIGATION OF SOLUTIONS

In this section, we will present some straightforward results derived from the free geodesic equation of structure as applied to the Randers metric with $\lambda = 0$ in some relatively simple cases. This is done primarily for the purpose of illustrating in relatively crude terms some immediate implications of the free geodesic theory, and not with the idea of reaching any comprehensive or rigorous conclusions. The following discussions will deal in turn with the application of the Randers metric equations to basic spherically symmetric and cosmological models.

4.1. The Spherically Symmetric Case

For purposes of this investigation, we will use polar coordinates represented by τ, ρ, θ, and ϕ, and define the metric elements as follows:

$$\begin{aligned} g_{\tau\tau} &= a(\rho) \\ g_{\rho\rho} &= b(\rho) \\ g_{\theta\theta} &= \rho^2 \\ g_{\phi\phi} &= \rho^2 \sin^2(\theta) \\ A_\tau &= f(\rho) \end{aligned} \tag{62}$$

The other variable in this model is K, which is also understood to be a function of ρ only.

The equation of structure as represented in (53) and (54) may now be expressed by the following four equations:

$$\left(\frac{\rho^2}{K^3\sqrt{ab}}\, f_{,\rho}\right)_{,\rho} = 0 \tag{63}$$

$$\left(\frac{\rho^2}{K^3\sqrt{ab}}\, (4f^2 + a)_{,\rho}\right)_{,\rho} = 0 \tag{64}$$

$$\left(\frac{1}{\rho} - \frac{3}{2}\frac{K_{,\rho}}{K}\right)\left(\frac{a_{,\rho}}{a} + \frac{b_{,\rho}}{b}\right) + 3\,\frac{K_{,\rho,\rho}}{K} = 0 \tag{65}$$

$$b - 1 + \frac{\rho}{2}\left(\frac{b_{,\rho}}{b} - \frac{a_{,\rho}}{a}\right) - 2\rho^2\,\frac{f_{,\rho}{}^2}{a} + 3\rho\,\frac{K_{,\rho}}{K} = 0 \tag{66}$$

Since our purpose is primarily to study the effects that the *K*-field might have in ordinary observable events, we will be content with formulating a polar approximation to a solution, restricted to asymptotically flat cases with appropriate choices for the limiting values of the variables. Taking the approximation to a second order of smallness in all variables, we have for sufficiently large ρ

$$f \approx q\,\frac{1}{\rho} + 3qk\,\frac{1}{\rho^2} \tag{67}$$

$$a \approx -1 + 2M\,\frac{1}{\rho} + (6Mk - 4q^2)\,\frac{1}{\rho^2} \tag{68}$$

$$b \approx 1 + (2M + 6k)\,\frac{1}{\rho} + (4M^2 + 27\,Mk + 42k^2 + 5q^2)\,\frac{1}{\rho^2} \tag{69}$$

$$K \approx 1 + k\,\frac{1}{\rho} + \left(Mk + \frac{7k^2}{2} + q^2\right)\frac{1}{\rho^2} \tag{70}$$

where *M*, *q*, and *k* are parameters which may be taken to represent a central mass, electric charge, and "*K* source" located at the origin.

In view of the preliminary conclusions drawn from the analysis referred to at the end of Section 3.7, we will generally assume that *k*, which represents a generator of a *K* field that is independent of the electromagnetic field, must be negligible or zero.

4.1.1. Units

For purposes of the following discussions we will adopt geometrized Planck units in which $G = c = \hbar = 1$. We will refer to these units as "natural" units. The natural units of length, time, mass, and charge are the Planck length, Planck time, Planck mass, and Planck charge:

$$l_{\mathrm{P}} = \sqrt{\frac{G\hbar}{c^3}} \approx 1.6 \times 10^{-33}\ \mathrm{cm} \tag{71}$$

$$t_{\mathrm{P}} = \frac{l_{\mathrm{P}}}{c} \approx 5.4 \times 10^{-44}\ \mathrm{sec} \tag{72}$$

$$m_{\mathrm{P}} = \frac{l_{\mathrm{P}}c^2}{G} \approx 2.2 \times 10^{-5}\ \mathrm{gm} \tag{73}$$

$$q_{\mathrm{P}} = \sqrt{\hbar c} \approx 5.6 \times 10^{-9}\ \mathrm{esu} \tag{74}$$

Because we have chosen to scale the equations of gravity in such a way that $g_{\tau\tau} \to -1$ and $g_{\rho\rho} \to 1$ in the asymptotically flat limit, the natural units will also serve as units for the time (τ) and radial (ρ) coordinates in that limit.

Measuring the charge of the electron in these units, we have

$$\frac{e}{\sqrt{\hbar c}} = \sqrt{\alpha} \approx 8.5 \times 10^{-2} \tag{75}$$

where α is the fine structure constant.

4.1.2. The Interaction Force

If $k = 0$, the most significant term in the interaction force defined in (61), from (67) and (70), is

$$\begin{aligned} T_{(E)i}{}^{m}{}_{;m}\rho^{i} &= -T_{(K)i}{}^{m}{}_{;m}\,\rho^{i} \\ &\approx \frac{q^4}{2\pi\rho^7} \end{aligned} \tag{76}$$

where ρ^i represents a unit vector in the direction of the ρ coordinate. By contrast, if we started with T_{ij} based directly on (57), defining one energy component as $(1/4\pi)(F_{im}F_j{}^m - g_{ij}F_{mn}F^{mn})$ and the other energy component as $(3/16\pi)(-2K_{,i;j}/K + g_{ij}g^{mn}K_{,m;n}/K)$, we would have an additional term in the interaction force:

$$\frac{3}{16\pi}\,(E_{mn}E^{mn})_{,i}\rho^{i} \approx \frac{3q^2}{4\pi\rho^5} \tag{77}$$

For sufficiently large ρ, this term would outweigh the interaction force found in (76) by a substantial factor. In the case of an electric charge equal to that of the electron, with no K source, at the Compton radius of the electron this factor would be $3\rho^2/2\alpha \approx 4.2 \times 10^{42}$. This supports the adoption of the formulation in (58)–(60), which eliminates the unnecessarily large interaction force term and gives us two nearly independent energy fields.

4.1.3. The Electromagnetic Field

In the absence of an independent K source, (67) shows that the extra term in (54) has no effect on the electromagnetic field up to the second order of polar terms. If we carry out the estimation to the third order, we find a term $\frac{5}{2}\,q^3/\rho^3$. This term is less than the first-order term by a factor of $\frac{2}{5}\,\rho^2/q^2$. For a charge equal to that of the electron, at the Compton radius this factor has a value of approximately 1.1×10^{42}. Clearly an effect of such small size would not be detected in ordinary laboratory experiments, and we may say that (54) appears consistent with observed reality.

4.1.4. The Gravitational Field

Looking at (68), we see that the electromagnetic charge affects $g_{\tau\tau}$ by a term $-4q^2/\rho^2$, as compared with $-q^2/\rho^2$ in the Reissner–Nordstrom solution. The increased value arises from the second-order influence of the electric charge on the K field, as reflected in (70), which affects $g_{\tau\tau}$ by way of the term $(3/8\pi)g_{ij}g^{mn}K_{,m;n}/K$ in (60). This does not appear to be inconsistent with any experimental evidence. Our knowledge of the effect of charged particles on the gravitational field is largely theoretical, and it does not seem unreasonable to assume that a significant part of it arises from sources other than the electromagnetic field itself. Taking account of the additional mass in the K field generated by the electromagnetic field would simply have the effect of increasing the theoretical radius of the electron by a factor of four over the classical Compton radius.

4.1.5. The K Energy Field

Restricting ourselves again to the situation where $k = 0$, we can determine the values of the energy fields surrounding a charged particle, as follows:

$$T_{(E)\tau\tau} \approx \frac{1}{8\pi}\frac{q^2}{\rho^4} \tag{78}$$

$$T_{(K)\tau\tau} \approx -\frac{3}{4\pi}\frac{q^2}{\rho^4} \tag{79}$$

$$T_{(E)\rho\rho} \approx -\frac{1}{8\pi}\frac{q^2}{\rho^4} \tag{80}$$

$$T_{(K)\rho\rho} \approx -\frac{3}{2\pi}\frac{q^2}{\rho^4} \tag{81}$$

$$T_{(E)\phi\phi} = T_{(E)\theta\theta}\sin^2(\theta) \approx \frac{1}{8\pi}\frac{q^2\sin^2(\theta)}{\rho^2} \tag{82}$$

$$T_{(K)\phi\phi} = T_{(K)\theta\theta}\sin^2(\theta) \approx \frac{3}{2\pi}\frac{q^2\sin^2(\theta)}{\rho^2} \tag{83}$$

These results are somewhat troubling in that they show that the K energy field in this case violates the weak energy condition. It is important to note, however, that the K energy field does satisfy the strong energy condition:

$$T_{(K)\tau\tau} - \tfrac{1}{2}g_{\tau\tau}g^{mn}T_{(K)mn} = \frac{3}{8\pi}\frac{q^2}{\rho^2} \tag{84}$$

Furthermore, this negative energy field can be viewed as something of a "phantom" because (1) its interaction with the electromagnetic field is extremely weak, as discussed in Section 4.2.2, and (2) the interaction between the K fields of two nearby particles should also be expected to be weak because the more significant components (in the spherically symmetric case) of the K energy, $(3/8\pi)(-K_{,i;j}/K + g_{ij}g^{mn}K_{,m;n}/K)$, are essentially additive for sufficiently large ρ, where $K_{,i;j} \ll K$. Given the fact that the weak energy condition is only a theoretical restriction, and that in practical terms the effects of the K field appear consistent with observed results, we do not regard the violation of the weak energy condition in this case to be a stumbling block to the consideration of generalized relativity as a theory applicable to our universe.

4.2. The Cosmological Case

A strictly homogeneous, isotropic solution of (53)–(54) can have no electromagnetic field, assuming compliance with the Lorentz gauge. The cosmological solution of these equations therefore resolves to the equations of the Robertson–Walker model with energy field limited to $T_{(K)ij}$, where K is a function of time only. Adopting the approach used in Wald (1984), we have

$$g_{\tau\tau} = -1 \tag{85}$$

$$g_{xy} = a^2(\tau)y_{xy} \tag{86}$$

where (1) τ is the time coordinate, (2) the xy subscripts refer to any pair of spatial coordinates, (3) the spatial coordinates are ψ (the radial coordinate, normalized to the size of the universe in the positively curved case), θ, and ϕ; (4) $y_{\psi\psi} = 1$, and (5) $y_{\theta\theta} = y_{\phi\phi}/\sin^2(\theta) =$ either ψ, $\sin(\psi)$, or $\sinh(\psi)$, depending on whether space is flat, positively curved, or negatively curved.

We now find that $T_{(K)ij}$ corresponds to a perfect fluid with density and pressure

$$\rho = -\frac{3}{16\pi}\frac{K_{,\tau,\tau}}{K} + \frac{9}{16\pi}\frac{a_{,\tau}K_{,\tau}}{aK} \tag{87}$$

$$P = -\frac{3}{16\pi}\frac{K_{,\tau,\tau}}{K} - \frac{3}{16\pi}\frac{a_{,\tau}K_{,\tau}}{aK} \tag{88}$$

The evolution equations are

$$3\,\frac{a_{,\tau}^2}{a^2} = 8\pi\rho - \frac{3k}{a^2} = -\frac{3}{2}\frac{K_{,\tau,\tau}}{K} + \frac{9}{2}\frac{a_{,\tau}K_{,\tau}}{aK} - \frac{3k}{a^2} \tag{89}$$

$$3\,\frac{a_{,\tau,\tau}}{a} = -4\pi(\rho + 3P) = 3\,\frac{K_{,\tau,\tau}}{K} \tag{90}$$

where k is a constant which is set equal to 0 in the case of flat space, 1 in the case of positively curved space, or -1 in the case of negatively curved space. (This k, of course, has nothing to do with our earlier K source k.)

Equation (90) is solved by either:

Case 1:

$$a = \alpha K \tag{91}$$

where α is a constant; or

Case 2:

$$K = \sqrt{\alpha / y_{,\tau}} \tag{92}$$

$$a = y\sqrt{\alpha / y_{,\tau}} \tag{93}$$

where α is a constant and y is an arbitrary smooth function of time. In either of these cases, we can substitute in (89) and have a single second- or third-degree differential equation in a single function of τ. Working out these solutions is beyond the scope of this paper. In the simplest case, however, which is Case 1 above in a flat universe, it is easy to see that we have

$$K = a = e^{\gamma\tau} \tag{94}$$

$$\rho = \frac{3}{8\pi}\gamma^2 \tag{95}$$

$$P = -\frac{3}{8\pi}\gamma^2 \tag{96}$$

where γ is an arbitrary constant and we have eliminated constant scale factors. [Note that although (94) indicates an expanding universe if $\gamma > 0$, the energy density and pressure of the K field remain constant, indicating an element of "continuous mass creation" in the K field in this model.]

It may be questioned whether the foregoing analysis has any real meaning, since the universe certainly contains energy derived from electromagnetic fields and other sources other than the K field. Such energy may originate from locally nonisotropic fields which are averaged out to a globally isotropic perfect fluid energy in the Robertson–Walker model. It is not beyond the realm of possibility, however, that the universe may contain a significant, or even dominant, isotropic K-energy field equivalent or comparable to that described in (94)–(96). In that case, we may have found another explanation for the calculated presence of dominant quantities of "dark matter" throughout the universe.

4.3. Beyond the Simple Cases

Our study of solutions has yielded some interesting conclusions in certain very simple and limited cases. Other cases will doubtless raise significant complexities. In particular, additional significant and possibly unpredictable effects may begin to appear from the influence of the K field where F_{ij} becomes very large, such as the region very close to a charged particle. In such a region, however, we would also have to consider, on the observational side, the influence of the strong and weak fields and other quantum effects, and, on the theoretical side, the possible additional effects that would arise from adopting a nonzero value for λ or from including additional elements of the geodesic function which are not included within the Randers-compatible model on which we have focused. A great deal more work is required, therefore, before we may begin to assess the applicability and predictive value of the free geodesic theory in such a realm.

5. CONCLUSION

Our study has shown that the basic conceptual premises of the free geodesic theory lead rather directly to a result which represents a significant expansion of the ability of relativity theory to account for the behavior of observed phenomena. A large field remains to be explored which may show further correspondences to additional aspects of observed reality. Further work in this field would appear called for on the basis of these initial results.

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